ON THE LOGICAL ANALYSIS OF THE FOUNDATIONS OF VECTOR CALCULUS

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ABSTRACT

A critical analysis of the foundations of standard vector calculus is proposed. The methodological basis of the analysis is the unity of formal logic and of rational dialectics. It is proved that the vector calculus is incorrect theory because: (a) it is not based on a correct methodological basis – the unity of formal logic and of rational dialectics; (b) it does not contain the correct definitions of “movement”, “direction” and “vector”; (c) it does not take into consideration the dimensions of physical quantities (i.e., number names, denominate numbers, concrete numbers), characterizing the concept of ”physical vector”, and, therefore, it has no natural-scientific meaning; (d) operations on “physical vectors” and the vector calculus propositions relating to the ”physical vectors” are contrary to formal logic.

Keywords: Mathematics, vector calculus, geometry, physics, engineering, philosophy of science

INTRODUCTION

As is well known, the mathematical formalism of vector calculus is widely and successfully used in natural sciences [1-7]. However, this does not mean that the problem of validity of vector calculus is now completely solved, or that the foundations of vector calculus are not in need of formal-logical analysis. In my view, standard vector calculus cannot be considered as absolute truth if there is no formal-logical substantiation of this calculus. Recently, there has arisen a necessity for critical analysis of the foundations of vector calculus. But there are no works devoted the analysis of vector calculus within the framework of the unity of formal logic and of rational dialectics. The purpose of the present work is to propose the correct analysis of the foundations of vector calculus. The analysis is carried out within the framework of the correct methodological basis: the unity of formal logic and of rational dialectics.

1. ANALYSIS OF THE CONCEPTS OF “DIRECTION” AND “VECTOR"

As is well known, in mathematics, physics, and engineering, a vector (or Euclidean vector, or geometric vector, or spatial vector) is called quantitative characteristics which has not only a numerical value, but also the direction [7, 8]. In other words, vector is a line segment with a definite direction (or graphically is an arrow), connecting an initial point with a terminal point. I.e., vector is a geometric object that has magnitude (or length) and direction and can be added to other vectors according to vector algebra. Physical examples of vector quantities are material point displacement, velocity and acceleration of a material point, as well as a force. Therefore, analysis of the concept of “vector” is not possible without the definitions of concepts of “movement” and “direction”.

1. Movement is a change in general, any interaction of material objects. Category of "movement" is a scientific concept that reflects the most common and essential property of phenomena (processes), the most common and essential relations and connections in reality. Movement is an attribute of matter. In accordance with the dialectical principle of the unity of matter and movement, the movement does not exist without material objects. But the movement is not a material object. The movement is manifested as the unity of opposites: changeableness and stability, continuity and discontinuity. Concretization of the movement is the main forms of movement: mechanical, physical (thermal, electromagnetic, gravitational, atomic, and nuclear), chemical, biological, informational, and social ones.

2. Change as a process can be of two types: a qualitative change and quantitative change. The qualitative change (i.e. a change of qualitative determinacy) is studied by dialectical logic and natural sciences. The quantitative change (i.e. a change of quantitative determinacy) within the limits of certain qualitative determinacy is studied by formal logic and mathematics. The quantitative change can be studied only within a reference system which contains a clock as component part.

3. A clock (i.e., a device containing a working clock mechanism, moving the arrow and the fixed dial) determines the time and time characterizes the clock. Time is a concrete concept because it expresses the property of the clock mechanism (clock process). Time t is a universal variable (with the dimension of "second"), an information basis that is used to put in order of information about events and processes in the world. Time t is defined by the following mathematical expression [9]: $t_n = n\tau$ where $n = 0, 1, 2, ...$; τ is elementary (unit) duration which can be made as small as desired. Concrete numbers (denominate numbers) t_n have one and the same qualitative determinacy (i.e., dimension of "time"). The set of numbers t_n forms an ordered sequence. A member of the sequence is called a moment of time. Numerical values of quantity t_n is changed due to clock mechanism which continuously changes numerical values of the quantity n.

4. The mechanical form of movement (in particular, the motion of a material point M) is studied in a reference system which represents the unity of the system of coordinates and clock. The system of coordinates is a system of measuring devices which determines the position (i.e., the set of coordinates) of a material point M in space. (For example, the Cartesian coordinate system represents the system of three connected measuring scales (drawing scales): straight lines Ox, Oy, Oz with printed concrete numbers (denominate numbers) having the identical dimension of "meter"). The space of the object (for example, geometric space, and energy space) is the set of possible (available) states of the material object (in particular, the set of positions of the material point M). Each state is characterized by a certain concrete number (denominate number) having a dimension. Movement of an object in space is a process of transition from some states to other states, i.e. the process of transition from some concrete (denominate) numbers to other concrete (denominate) numbers.

5. A process has the beginning (i.e., the beginning of the changes) and the end (i.e., the end of the changes). The transition from the initial state to the final state represents the sum of elementary transitions and, therefore, is characterized by an increase of the changes. In other words, the total change is the sum of elementary changes. Since elementary change is characterized by the concrete (denominate) number having a dimension, the total change has dimension as well and is expressed by the following mathematical formula:

$$s_n = n\lambda,$$

where $n = 0, 1, 2, 3, \ldots$, λ is elementary (single) change which has the dimension and is assumed to be constant. The set of denominate numbers s_n forms an ordered sequence. The numerical values of the denominate quantity s_n are changed if the numerical values of the quantity n are changed. If the numerical values of the quantity n are not changed with time, the process is not realized.

6. A process is characterized by the direction (directivity) of change, the rate of change, and acceleration of change. If the process is not realized, the direction (directivity), the rate, and acceleration do not exist. Explanation is that the direction (directivity), rate, and acceleration are the properties of the process and not the properties of the material object. Therefore, the direction of change determines the order of the number set but an ordered number set does not determine the direction. Neither pure mathematics nor applied mathematics (i.e., the mathematical formalism of the natural sciences) does not contain a mathematical (calculation) process because the mathematics does not represent a computer or some other material device that realizes the process of change of the values of the quantity s_n. Change of the valuesof the quantity n is carried out by an operator (person). Therefore, the correct mathematical formalism cannot contain the concepts of "direction (directivity)" and "vector".

7. If one assumes that the mathematical formalism contains the concepts "direction (directivity)" and "vector", then the formula for the quantity s_n to be written in the following vector form:

$$\vec{s}_n = n\vec{\lambda}$$

where $\vec{\lambda}$ is elementary (unit) vector. But since the numerical values of quantity n in this formula are not changed with time, the process of change of the numerical values of the quantity s_n in mathematical formalism is not realized. Therefore, this formula does not describe direction, and the mathematical formalism does not contain the concepts of "direction (directivity)", "vector", and "unit vector".

Thus, the "direction (directivity)" and "vector" are not mathematical objects (concepts). The concepts of "direction" and "vector" do not correspond to any geometric object (for example, a line segment). Indication of the boundary points of the line segment and designation of these points with the help of terms (words) "beginning" ("initial point") and "end" ("terminal point") do not define mathematically a geometric vector (because the order of points do not define the direction of movement). All points of the line segment have one and the same qualitative determinacy: concept of "initial point" and concept of "terminal point" are identical ones. Therefore, the terms "beginning" and "end" of the segment are not mathematical definitions of the concept of "direction (directivity)". An arrow is a visual (graphic) image of course. In other words, verbal, literal, symbolic, numerical, and graphical representations (display) of the beginning and the end of the segment are not a mathematical definition of the concept of "direction (directivity)". Therefore, the correct mathematical formalism can not and must not contain the concepts of "direction (directivity)" and "vector". The coordinate system represents a system of three connected drawing scales: straight lines Ox, Oy, Oz, which cannot be attributed to the direction. Also, straight lines Ox, Oy, Oz cannot contain the unit vectors. From the point of view of formal logic, the terms "direction (directivity)" and "vector" in mathematics and theoretical physics mean representation, i.e. imaginary image of the process, which is depicted with the help of an arrow.

2. ANALYSIS OF THE STANDARD PROPOSITIONS OF VECTOR ALGEBRA

The mathematical concept of "vector in generall" cannot be used in the natural sciences: this concept does not make sense in the natural sciences. The concept of "vector" as used in the natural sciences is characterized by the concepts of "denominate quantity" and "dimension of quantity". Therefore, analysis of the standard propositions of vector algebra must be done from this viewpoint.

1. As is known, the position of a vector in the Cartesian coordinate system $Oxyz$ is determined by its projections. The projection of the vector on the axis is defined as follows. There are vector $\vec{V}$ whose module has the dimension (for example, the dimension of speed, acceleration dimension, the dimension of power) and the axis Ox with the denominate numbers which have dimension of length. The projection of the vector $\vec{V}$ on the axis Ox is called the quantity (module, length) $\left|\vec{V}_x\right|$ of directed segment $\vec{V}_x$ located (placed, putted) on the axis Ox:

$$\left|\vec{V}_x\right| = \left|\vec{V}\right| \cos\alpha$$

where α is the angle between the vector $\vec{V}$ and the axis Ox. In general case, the vector $\vec{V}$ is decomposed into components under the Cartesian orthonormal basis $\vec{i}$, $\vec{j}$, $\vec{k}$ as follows:

$$\vec{V} = \left|\vec{V}_x\right|\vec{i} + \left|\vec{V}_y\right|\vec{j} + \left|\vec{V}_z\right|\vec{k}$$

where $\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit vectors of the Cartesian coordinate system; $\left|\vec{V}_x\right|$, $\left|\vec{V}_y\right|$, $\left|\vec{V}_z\right|$ are the projections of the vector on the corresponding axes. The principal importance of the basis $\vec{i}$, $\vec{j}$, $\vec{k}$ is that the linear operations on vectors under the given basis become the usual linear operations on numbers – the coordinates of these vectors. In my opinion, these standard expressions are not free from objection. The objection is that the standard expressions are contrary to the formal-logical laws.

Really, the standard expressions assert that segment $\left|\vec{V}_x\right|$ lies on the axis Ox (i.e., segment $\left|\vec{V}_x\right|$ coincides with the segment of axis Ox). From the point of view of formal-logical law of identity, this implies that these segments have the same qualitative determinacy (i.e., the same sense, the same dimensions):

(qualitative determinacy of the segment $\left|\vec{V}_x\right|$) =

(qualitative determinacy of the segment of the axis Ox).

But the segment $\left|\vec{V}_x\right|$ cannot lie on the axis Ox (i.e., the segment $\left|\vec{V}_x\right|$ cannot be coincided with a segment of the axis Ox) because these segments have different dimensions and, therefore, different qualitative determinacy (i.e., different senses). This statement is expressed by formal-logical law of absence of contradiction:

$$\textit{(qualitative determinacy of the segment } |\vec{V}_x|) \neq$$
$$\textit{(qualitative determinacy of the segment of the axis } Ox).$$

Consequently, the mathematical operation of finding the projection of the vector $\vec{V}$ on the coordinate axes represents the formal-logical error: violation of the law of absence of contradiction.

2. As is known, the rule of addition of vectors having the same qualitative determinacy is called the "triangle rule" or "parallelogram rule". Standard operation of addition of two vectors is defined as follows: the sum $\vec{V}_1 + \vec{V}_2$ of two vectors $\vec{V}_1$ and $\vec{V}_2$ is called the vector running from the beginning of the vector $\vec{V}_1$ to the end of the vector $\vec{V}_2$ under the condition that the vector $\vec{V}_2$ is applied to the end of the vector $\vec{V}_1$. Under the addition of two vectors, their projections on an arbitrary axis are added, and under the multiplication of a vector by any number, its projection on an arbitrary axis is multiplied by this number. In my view, these standard assertions are not free from objection. The objection is that the standard assertions are contrary to the formal-logical laws. Really, segments of vectors and segments of arbitrary axis have different qualitative determinacy (i.e., different senses). This implies that the segments of the vectors $\vec{V}_1$ and $\vec{V}_2$ cannot lie on a segment of an arbitrary axis (i.e., the segments of the vectors cannot coincide with a segment of an arbitrary axis). From the point of view of formal-logical law of identity, these segments can be coincided if only they have identical qualitative determinacy (i.e., the same dimension, the same meaning).

3. As is known, the scalar product of two vectors $\vec{V}$ and $\vec{F}$ is defined as follows: (a) one brings the initial points of vectors in coincidence with each other (i.e., the initial points are connected); (b) one postulates the relation

$$\vec{V} \bullet \vec{F} = |\vec{V}||\vec{F}| \cos\varphi$$

where the point between symbols of vectors denotes the operation of scalar multiplication of vectors, φ is angle between the vectors. The expression $|\vec{V}|\cos\varphi$ represents a denominate number: the projection of the vector $\vec{V}$ on the vector $\vec{F}$. Also, the expression $|\vec{F}|\cos\varphi$ represents a denominate number: the projection of the vector $\vec{F}$ on the vector $\vec{V}$. In my opinion, the standard definition of the scalar product of vectors is not free from objection. The objection is that the standard definition is contrary to the formal-logical laws.

Really, the coincidence (connection) of initial points of vectors and the formation of the projections imply that the dimension of length (i.e., the qualitative determinacy) of the vector $\vec{V}$ is identical to the dimension of length (i.e., the qualitative determinacy) of the vector $\vec{F}$:

$$\textit{(qualitative determinacy of the vector } \vec{V}) =$$
$$\textit{(qualitative determinacy of the vector } \vec{F}).$$

In general case, however, the dimensions of vector lengths are different. Therefore, these vectors cannot have a common point, and the multiplication can not be performed (i.e. the multiplication has no sense). This fact is expressed formal-logical law of absence of contradiction:

(qualitative determinacy of the vector $\vec{V}$ *)* $\neq$
(qualitative determinacy of the vector $\vec{F}$ *).*

Consequently, the mathematical operation of scalar product of two vectors represents a formal-logical error: a violation of the law of absence of contradiction.

4. As is known, the cross-product of two vectors $\vec{V}$ and $\vec{F}$ is defined as follows: (a) one brings the initial points of vectors in coincidence with each other (i.e., the initial points are connected); (b) one postulates the relation

$$\vec{H} \equiv \vec{V} \times \vec{F} = \vec{h}\,|\vec{V}|\,|\vec{F}|\,|\sin\varphi|$$

where the cross between the symbols of vectors denotes the operation of vector multiplication of vectors, φ is angle between the vectors, $\vec{H}$ is vector which is normal to the plane formed by the vectors $\vec{V}$ and $\vec{F}$; $\vec{h}$ is unit vector which is normal to the plane. Under the established agreement, the direction of vectors $\vec{H}$ and $\vec{h}$ is determined by the “right-hand screw rule”. In my opinion, the standard definition of the cross-product of vectors is not free from objection. The objection is that the standard definition is contrary to the formal-logical laws. Really, the coincidence of the initial points of the three vectors means that the dimensions of lengths (i.e., the qualitative determinacy) of the vectors $\vec{V}$, $\vec{F}$, and $\vec{H}$ are identical:

(qualitative determination of the vector $\vec{V}$ *)* $=$
(qualitative determination of the vector $\vec{F}$ *)* $=$
(qualitative determination of the vector $\vec{H}$ *).*

In general case, however, the dimensions of the lengths of the vectors are different. Therefore, these vectors cannot have a common point, and the operation of vector multiplication cannot be performed (i.e., the operation of multiplication has no sense). This fact is expressed formal-logical law of absence of contradiction:

(qualitative determinacy of the vector $\vec{V}$ *)* $\neq$
(qualitative determinacy of the vector $\vec{F}$ *)* $\neq$
(qualitative determinacy of the vector $\vec{H}$ *).*

Consequently, the mathematical operation of the cross-product of vectors is a formal-logical error: a violation of the law of absence of contradiction.

DISCUSSION

1. As is known, the confidence in the scientific method of research and in rational thinking replaced all other ways of cognition in the 20th century. Rational thinking represents the greatest achievement of mankind. Rationalization of thinking and of science is dialectical

imperative of our time. The development of rational thinking in the 21st century leads to critical analysis, reconsideration, and rationalization of the generally accepted theories created by the classics of science (for example, N. Bohr, E. Schrödinger, W. Heisenberg, A. Einstein, I. Newton, G. Leibniz, L. Euler, J. Lagrange, A. Cauchy, W.R. Hamilton, J.W. Gibbs, O. Heaviside, etc.). Rationalization and critical analysis of science are two side pieces (component factors) in progress of science. Critical analysis and rationalization of theories are based on formal-logical analysis of scientific concepts, of the completeness of concepts, of the completeness of a system of concepts because "only the completeness leads to clarity" (Confucius). Recently, independent researchers give attention to critical analysis of theoretical physics, mathematics, biology, etc. (see, for example, www.gsjournal.net). In the process of critical analysis and of interpretation of scientific theories, "...we can hardly rely on any of the old principles even if they are very common. The only mandatory requirement is the absence of logical contradictions." (N. Bohr). Logical consistency of theories is achieved with use of the formal-logical laws. And a natural-scientific interpretation of theories is based on the use of rational dialectics. The system of universal (general-scientific) concepts and laws – i.e., science of the general laws of development of the Nature, human society, and correct thinking – is the unity of formal logic and rational dialectics. This unity is not only correct methodological basis of science but also the correct methodological basis for a critical analysis of theories.

2. The origin of vector calculus is closely related to the needs of mechanics and physics: the idea of motion, the concepts of process, velocity, acceleration, displacement, force, and vector were introduced into mathematics in the 17-18th centuries. The modern meaning of the word "vector" represents generalization of its previous (out-of-date) meaning in astronomy, where, in 18th century, a vector is called an imaginary straight line segment connecting the planet to the center (focus) of the motion. At present, vector calculus is a branch of mathematics in which one studies the properties of operations on vectors. But the mathematical formalism does not contain motion, mathematical process. A mathematical process is carried out only in computers. (This is why continual mathematics must be replaced by discrete mathematics – computer mathematics).

In specific scientific problem, one considers the quantities of the various natures. These quantities have different dimensions: length, area, volume, weight, temperature, speed, strength, etc.). If one selects a (define, explicit, appointed) determined unit, then each value of the quantity must be expressed by denominate number. But mathematics does not consider the specific quantities: the mathematical propositions and laws are formulated, abstracting from the specific nature of the quantities, taking into consideration only their numerical values. In line with this, mathematics considers the quantity in general, the vector in general, and so on, neglecting the natural-scientific meaning of the quantity.

Abstract mathematical propositions, theories, and models cannot be tested and used in the natural sciences. From the point of view of formal logic and of rational dialectics, in order to test and use mathematical propositions, theories, and models in practice, it is necessary to define the natural-scientific (practical) meaning of mathematical concepts (objects) and relations, i.e., to consider not a "quantity in general", a "number in general", "vector in general", but to consider the nature (i.e., dimensions) of quantities (length, area, volume, weight, temperature, speed, acceleration, displacement, force, etc.). From this point of view, the standard vector calculus does not have a natural-scientific meaning because the standard vector calculus is based on the concept of "vector in general". Clarification of natural-scientific meaning of concept of "vector" and a logical analysis of operations on the "physical vectors" show that the standard propositions of vector calculus, relating to the "physical vectors", are contrary to formal logic.

3. There are two opinions about the existence of logical errors in generally accepted theories (for example, in physics and mathematics). The first opinion is that, although a theory (for example, the special theory of relativity) contains logical errors, "it works well" (Gerard't Hooft). The second opinion is that the system of four fundamental formal-logical laws is incomplete and insufficient for a panchreston (i.e., for complete explanation) and mathematical description of reality. In essence, these opinions are identical. However, in my opinion, these views are not free from objection. The objection is as follows. If one will discover additional formal logic laws, then the complete system of laws should not be contradictory: the four basic laws will retain its place and importance in a new, complete system (in other words, the four basic laws will not be refuted). In this case, the theories that are erroneous in "incomplete" logical system will also be erroneous in the "complete" logical system. And the theories that contain logical errors are false in essence. But the following questions will always remain open: Why devices that are based on false scientific theories (ideas) work? Why do the false scientific theories contribute to the development of mankind? Where is the limit of development based on false theories? What is the danger of development based on false theories? What are the essence and predestination of development?

CONCLUSION

Thus, the formal-logical and dialectical analysis of the foundations of vector calculus leads to the following main results: the standard vector calculus is incorrect theory because

a. it is not based on the correct methodological basis: the unity of formal logic and of rational dialectics;
b. it does not contain the correct definitions of concepts of "movement", "direction", and "vector";
c. it does not take into consideration the dimensions of physical quantities (i.e., number names, denominate numbers, concrete numbers), characterizing the concept of "physical vector", and, therefore, it has no natural-scientific meaning;
d. operations on "physical vectors" and the theoretical propositions of the standard vector calculus, relating to the "physical vectors", are contrary to formal logic.

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